Indian J. Phys. 47, 546–554, (1973)

Fluctuating flow of a viscoelastic fluid past an infinite flat plate with uniform suction

S Bhattacharjee

Department of Mathematics, Banaras Hindu University Varanasi-5

(*Received* 2 *August* 1972)

The two dimensional incompressible viscoelastic fluid flow problem along an infinite flat plate is discussed when the suction vlocity normal to the plate is uniform and the external flow velocity varies periodically with time Expressions for velocity and skin-friction are obtained in dimensionless forms and the effect of fluid's elastic parameter on their behaviour is sought.

1. Introduction

Lighthill (1954) first studied the effect of unsteady fluctuations in the external flow velocity on the boundary layer flow. Thereafter, some interesting features have been found by Stuart (1955) for an oscillatory flow over an infinite plate with uniform suction. Recently Rath & Mishra (1969) have discussed the exponential boundary layer flow of a second order fluid along a porous infinite flat plate with uniform suction. They have shown that the steady components of the velocity and skin-friction coefficients are influenced by the elastic parameter of the fluid, but their unsteady parts are not affected by this parameter, and therefore, the manner of unsteadiness is not different from the Newtonian behaviour.

In the present work an attempt has been made to study the fluctuating flow of a viscoelastic fluid past an infinite flat plate with uniform suction. The effect of fluid's elastic parameter on the skin-friction amplitude and the velocity field is also discussed. Here the viscoelastic fluid considered is of 'Kuvshinski type' (Michael & Bird 1962). The external flow velocity has been taken as $U_0(1+\epsilon \exp(int))$ and v_0 is a non-zero negative constant suction velocity.

2. Equations of Motion

The equations governing the viscoelastic fluid model considered here consist of stress-strain rate law

$$\left(1+\lambda_0 \frac{D}{Dt}\right) p_{ik} = 2\mu e_{ik}, \qquad \ldots \quad (2.1)$$

where $$\frac{D}{Dt}(p_{ij}) = \frac{\partial p_{ij}}{\partial t} + v_m \frac{\partial p_{ij}}{\partial x_m}; \qquad \ldots \quad (2.2)$$

and $$e_{ij} = 1/2\left[\left(\frac{\partial v_i}{\partial x_j}\right)+\left(\frac{\partial v_j}{\partial x_i}\right)\right]. \tag{2.3}$$

The physical significance of λ_0 (the relaxation time) is that, if the motion suddenly stops, the shear stress will decay as $\exp(-t/\lambda_0)$.

Here, the stress tensor is given by

$$s_{ij} = -p\partial_{ij}+p_{ij}, \tag{2.4}$$

where p is the static pressure, ∂_{ij} is the Kronecker delta and p_{ij} is a tensor usually related to the rate of strain, e_{ij}, by the equation of state (2.1).

We consider a two dimensional incompressible viscoelastic fluid flow problem along an infinite plane porous wall The flow is independent of the distance parallel to the wall and the suction velocity v, normal to the wall, is directed towards it and is constant. The axis of x is taken along the wall and the axis of y normal to the wall. Then the equations governing the flow are

$$\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}\right) = \frac{\partial p}{\partial x}+\frac{\partial p_{xy}}{\partial y}, \tag{2.5}$$

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}+\frac{\partial p_{yy}}{\partial y}, \tag{2.6}$$

$$\frac{\partial v}{\partial y} = 0 \tag{2.7}$$

Although $\partial v/\partial y = 0$ in (2.7) shows that the velocity v is a function of time only, we now further restrict consideration to the case of v equal to a negative constant $(-v_0)$. Equation (2.1) then gives

$$p_{xx}+\lambda_0\left(\frac{\partial p_{xx}}{\partial t}-v_0\frac{\partial p_{xx}}{\partial y}\right) = 0, \tag{2.8}$$

$$p_{xy}+\lambda_0\left(\frac{\partial p_{xy}}{\partial t}-v_0\frac{\partial p_{xy}}{\partial y}\right) = \mu\frac{\partial u}{\partial y}, \tag{2.9}$$

$$p_{yy}+\lambda_0\left(\frac{\partial p_{yy}}{\partial t}-v_0\frac{\partial p_{yy}}{\partial y}\right) = 0, \tag{2.10}$$

Clearly, $p_{xx} = 0$, $p_{yy} = 0$, are particular solutions of the equations (2.8) and (2.10) respectively. Putting $p_{yy} = 0$ in (2.6) we get

$$\frac{\partial p}{\partial y} = 0, \tag{2.11}$$

from which it follows that p is independent of y. Now let us suppose $-(\partial p/\partial x)$ is equal to $\rho(dU/dt)$, for the motion outside the boundary layer. Here U is the external flow velocity. Equation (2.5) is then reduced to

$$\rho\left(\frac{\partial u}{\partial t}-v_0\frac{\partial u}{\partial y}\right) = \rho\frac{dU}{dt}+\frac{\partial p_{xy}}{\partial y}. \qquad \ldots \quad (2.12)$$

Eliminating p_{xy} in equations (2.9) and (2.12), we get

$$\frac{\partial u}{\partial t}-v_0\frac{\partial u}{\partial y}+\lambda_0\frac{\partial^2 u}{\partial t^2}-2\lambda_0 v_0\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)$$

$$=\frac{dU}{dt}+\lambda_0\frac{d^2U}{dt^2}+(\nu-\lambda_0 v_0^2)\frac{\partial^2 u}{\partial y^2} \qquad \ldots \quad (2.13)$$

where $\nu = \mu/\rho$(kinematic viscosity).

This equation is subject to the conditions $u = 0$ at $y = 0$ and $u \to U(t)$ as $y \to \infty$, $U(t)$ being the velocity outside the boundary layer.

We now introduce non-dimensional quantities defined by

$$\bar{y} = \frac{yv_0}{\nu}, \quad \bar{t} = \frac{v_0^2 t}{4\nu}, \quad \bar{n} = \frac{4\nu n}{v_0^2}, \quad \bar{u} = \frac{u}{U_0},$$

$$\bar{U} = \frac{U}{U_0}, \quad \text{and} \quad \lambda = \frac{\lambda_0 v_0^2}{\nu} \text{ (elastic parameter)} \qquad (2.14)$$

where U_0 is reference velocity and n is the frequency.

Equation (2.13) then becomes

$$\frac{1}{4}\cdot\frac{\partial \bar{u}}{\partial \bar{t}}-\frac{\partial \bar{u}}{\partial \bar{y}}+\frac{\lambda}{16}\frac{\partial^2 \bar{u}}{\partial \bar{t}^2}-\frac{\lambda}{2}\frac{\partial}{\partial \bar{y}}\left(\frac{\partial \bar{u}}{\partial \bar{t}}\right)$$

$$=\frac{1}{4}\frac{d\bar{U}}{d\bar{t}}+\frac{\lambda}{16}\frac{d^2\bar{U}}{d\bar{t}^2}+(1-\lambda)\frac{\partial^2\bar{u}}{\partial \bar{y}^2} \qquad (2.15)$$

subject to the condition,

$$\bar{u} = 0 \quad \text{at} \quad \bar{y} = 0,$$

$$\bar{u} \to U \quad \text{as} \quad \bar{y} \to \infty. \qquad (2.16)$$

Let us suppose the periodic velocity of the form

$$\bar{U} = 1+\epsilon \exp(i\bar{n}\bar{t})$$

and

$$\bar{u} = f_1(\bar{y})+\epsilon \exp(i\bar{n}\bar{t}) f_2(\bar{y}). \qquad (2.17)$$

On dropping the bars we have

$$U = 1+\epsilon \exp(int),$$

and

$$u = f_1(y)+\epsilon \exp(int) f_2(y). \qquad (2.18)$$

Substituting (2.17) in (2.15) and comparing harmonic terms and on dropping the bars, we get

$$(1-\lambda)f_1''(y)+f_1'(y) = 0 \tag{2.19}$$

$$(1-\lambda)f_2''(y)+\left(1+\frac{\lambda in}{2}\right)f_2'(y)-\frac{in}{4}\left(1+\frac{\lambda in}{4}\right)f_2(y)$$

$$= -\frac{in}{4}\left(1+\frac{\lambda\, in}{4}\right) \tag{2.20}$$

subject to the condition

$$\left.\begin{aligned} &f_1 = f_2 = 0 \quad \text{at} \quad y = 0 \\ \text{and} \quad &f_1, f_2 \to 1 \quad \text{as} \quad y \to \infty. \end{aligned}\right\} \tag{2.21}$$

Here, dashes denote differentiation with respect to y.

The solutions of (2.19) and (2.20), satisfying (2.21) are

$$f_1 = 1-\exp\left(-\frac{y}{1-\lambda}\right) \tag{2.22}$$

$$f_2 = 1-\exp(-hy). \tag{2.23}$$

It can be noticed from the equations (2.22) and (2.23) that $\lambda < 1$, otherwise if $\lambda > 1$, it will cause a back flow in fluid motion. Hence the velocity field in the boundary layer is given by

$$u(y, t) = 1-\exp\left(-\frac{y}{1-\lambda}\right)+\epsilon\exp(int)(1-\exp(-hy)) \tag{2.24}$$

and the non-dimensional skin friction τ_ω is given by

$$\tau_\omega = \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{1}{1-\lambda}+\epsilon h\exp(int) \tag{2.25}$$

where

$$h = \frac{\left(1+\frac{\lambda\, in}{2}\right)+\left[\left(1-\frac{\lambda n^2}{4}\right)+in\right]^{\frac{1}{2}}}{2(1-\lambda)} \tag{2.26}$$

Here the real parts are only significant. Since h involves the square root of a complex number, it has two values, *viz.*, one with the real part positive and the other with the real part negative. Here we will consider only the real part positive.

If $$\left[\left(1-\frac{\lambda n^2}{4}\right)+in\right]^{\frac{1}{2}} = a+ib, \quad a > 0,$$

where $$a = \left[\frac{(1-\lambda n^2/4)+\{(1-\lambda n^2/4)^2+n^2\}^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}$$

and $$b = \left[\frac{-(1-\lambda n^2/4)+\{(1-\lambda n^2/4)^2+n^2\}^{\frac{1}{2}}}{2} \right]^{\frac{1}{2}}$$

then the solution of (2 20) will be of the form

$$f_2 = 1+A \exp\left(-1-a-\frac{\lambda in}{2}-ib\right)y$$

$$+B \exp\left(-1+a-\frac{\lambda in}{2}+ib\right)y. \tag{2.27}$$

From this and the boundary conditions (2.21), one can obtain (2.23) only if $a > 1$ In the other case, *viz.*, $0 < a < 1$, the problem becomes indeterminate. Furthermore, since $a > 1$, we will not take the value of f_2 in the equation (2.27) due to real part negative, *i.e.* for $f_2 = 1+B\exp\left(-1+a-\frac{\lambda in}{2}+ib\right)$, since $f_2 \to 1$ as $y \to \infty$

It is observed from the expressions for the velocity and skin-friction in (2.24) and (2.25) that their steady parts are also influenced by elastic parameter. Fvom from (2.24(we get

$$u(y, t) = 1-\exp\left(-\frac{y}{1-\lambda}\right)+\epsilon[M_r \cos nt - M_i \sin nt] \quad \ldots \quad (2.28)$$

where

$$M_r = 1-\exp(-h_r y)\cos h_i y, \quad M_i = \exp(-h_r y)\sin h_i y,$$

$$h_r = \frac{1+\left[\frac{R+(1-\lambda n^2/4)}{2}\right]^{\frac{1}{2}}}{2(1-\lambda)},$$

$$h_i = \frac{\lambda n/2+\left[\frac{R-(1-\lambda n^2/4)}{2}\right]^{\frac{1}{2}}}{2(1-\lambda)},$$

and $$R = \left[\left(1-\frac{\lambda n^2}{4}\right)^2+n^2\right]^{\frac{1}{2}}.$$

Also from (2.25) we get

$$\tau_\omega = \frac{1}{(1-\lambda)}+\frac{\epsilon}{(1-\lambda)}|B|\cos(nt+\alpha). \quad \ldots \quad (2.29)$$

where $$|B| = \frac{1}{2}\left[R+1+\frac{\lambda^2 n^2}{4}+\sqrt{2}\{R+(1-\lambda n^2/4)\}^{\frac{1}{2}} + \frac{\lambda n}{\sqrt{2}}\{R-(1-\lambda n^2/4)\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

and
$$\tan \alpha = \frac{\frac{\lambda n}{2}+\frac{1}{\sqrt{2}}\left[R-\left(1-\frac{\lambda n^2}{4}\right)\right]^{\frac{1}{2}}}{1+\frac{1}{\sqrt{2}}\left[R+\left(1-\frac{\lambda n^2}{4}\right)\right]^{\frac{1}{2}}}$$

Now for the small values of the frequency parameter n, we have

$$u(y,t) = 1-\exp\left(\frac{-y}{1-\lambda}\right)+\epsilon\exp(int)\left[1-\exp\left(-\frac{y}{1-\lambda}\right)\right.$$
$$+\frac{i(1+\lambda)}{4(1-\lambda)}ny\exp\left(-\frac{y}{1-\lambda}\right)+\left\{y+\frac{1(1+\lambda)^2}{2(1-\lambda)^2}y^2\right\}$$
$$\left.\times\frac{n^2}{16}\exp\left(-\frac{y}{1-\lambda}\right)\right]. \quad \ldots \quad (2.30)$$

and the skin-friction is given by

$$\tau_\omega = \frac{1}{(1-\lambda)}+\frac{\epsilon}{(1-\lambda)}\left[1+\frac{n^2}{16}(\lambda^2+3)\right]^{\frac{1}{2}}\cos\left[nt+\tan^{-1}\left\{\frac{n}{4}(1+\lambda)\right\}\right] \quad \ldots \quad (2.31)$$

For large values of the frequency, we have

$$u(y,t) = 1-\exp\left(\frac{-y}{1-\lambda}\right)+\epsilon\exp(int)\left[1-\exp\left\{\frac{-\frac{\lambda in}{2}+\left(in-\frac{\lambda n^2}{4}\right)^{\frac{1}{2}}}{2(1-\lambda)}y\right\}\right], \quad \ldots \quad (2.32)$$

and

$$\tau_\omega = \frac{1}{(1-\lambda)}+\frac{\epsilon}{2(1-\lambda)}\left[1+\frac{\lambda^2 n^2}{4}+\sqrt{L}+\frac{1}{2\sqrt{L}}+2H_r+\lambda n H_i\right]^{\frac{1}{2}}$$
$$\times\cos\left[nt+\tan^{-1}\left\{\frac{\frac{\lambda n}{2}+\frac{1}{\sqrt{2}}\left\{\sqrt{L}+\frac{1}{2\sqrt{L}}-\left(1-\frac{\lambda n^2}{4}\right)\right\}^{\frac{1}{2}}}{1+\frac{1}{\sqrt{2}}\left\{\sqrt{L}+\frac{1}{2\sqrt{L}}+\left(1-\frac{\lambda n^2}{4}\right)\right\}^{\frac{1}{2}}}\right\}\right] \quad \ldots \quad (2.33)$$

where
$$H_r = \frac{1}{\sqrt{2}}\left[\sqrt{L}+\frac{1}{2\sqrt{L}}+\left(1-\frac{\lambda n^2}{4}\right)\right]^{\frac{1}{2}}$$
$$H_i = \frac{1}{\sqrt{2}}\left[\sqrt{L}+\frac{1}{2\sqrt{L}}-\left(1-\frac{\lambda n^2}{4}\right)\right]^{\frac{1}{2}}$$

and
$$L = n^2-\frac{\lambda n^2}{2}+\frac{\lambda^2 n^4}{16}$$

We see from (2.29) that skin-friction has a phase lead over the velocity fluctuations at large distance from the plate and this phase lead increases with increasing n and λ which is also clear from the following figure.

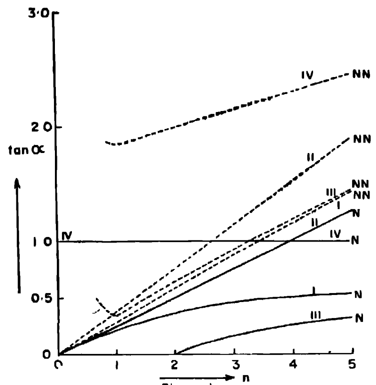

Figure 1

Skin-friction phase lead against frequency parameter n.

I Exact frequency approx
II Low ,, ,, ⟶ N : Newtonian ($\lambda=0$)
III : High ,, ,, ---→ NN. Non-Newtonian ($\lambda=\sqrt{2}$)
IV High (Rough) ,, ,,

High Frequency Approximation

$$\tan\,\alpha = \frac{\lambda n/2 + H_i}{1 + H_r}$$

High (Rough) Frequency Approximation

$$\tan\,\alpha = \frac{\dfrac{\lambda n}{2} + \dfrac{1}{\sqrt{2}}\left\{\dfrac{\lambda n^2}{4} + \left(\dfrac{\lambda^2 n^4}{16} + n^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}{\dfrac{1}{\sqrt{2}}\left\{-\dfrac{\lambda n^2}{4} + \left(\dfrac{\lambda^2 n^4}{16} + n^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}$$

Figure 2 shows that the amplitude of the skin-friction is also increased with increasing n and λ. An increase in skin-friction has also been predicted by Beard & Walters (1964) for the case of boundary layer flow of viscoelastic fluids near a stagnation point. Figure 2 (for Newtonian and viscoelastic fluids) also show a fair agreement between the exact amplitude $|h|$ and low frequency approximation upto about $n = 1$, though above $n = 1$, high frequency approximation is in rather more satisfactory agreement with the exact value. Only the high (rough) frequency approximation is not in agreement with the exact value.

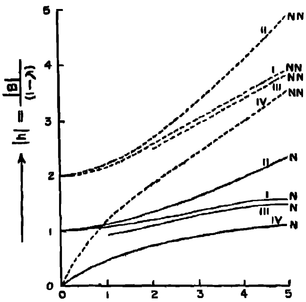

Figure 2
Skin-friction amplitude against frequency parameter n

I Exact frequency approx
II Low " "
III High " "
IV High (Rough) " "

——→ N Newtonian ($\lambda=0$)
----→ NN Non-Newtonian ($\lambda=1/2$)

High Frequency Approximation

$$|B| = \tfrac{1}{2}\left[1+\sqrt{L}+\frac{1}{2\sqrt{L}}+\frac{\lambda^2n^2}{4}+\left\{2\left(1-\frac{\lambda n^2}{4}+\sqrt{L}+\frac{1}{2\sqrt{L}}\right)\right\}^{\frac{1}{2}}\right.$$

$$\left.+\frac{\lambda n}{\sqrt{2}}\left(-1+\frac{\lambda n^2}{4}+\sqrt{L}+\frac{1}{2\sqrt{L}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

High (Rough) Frequency Approximatin

$$|\mathrm{B}| = \tfrac{1}{2}\left[\frac{\lambda^2n^2}{4}+\left(\frac{\lambda^2n^4}{16}+n^2\right)^{\frac{1}{2}}+\frac{\lambda n}{\sqrt{2}}\left\{\frac{\lambda n^2}{4}+\left(\frac{\lambda^2n^4}{16}+n^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

Figure 3 for the fluctuating parts of velocity profile M_r and M_i against the distance from the wall for $\lambda = 0$, $\lambda = 0.5$, and $n = 1$ shows that M_r increases with increasing y while M_i decreases From this figure it is also obvious that M_r for viscoelastic fluids is more than M_r for Newtonian fluids, whereas the reverse is true for M_i.

In figure 4, the transient velocity profile $u = 1-\exp(-y/1(-\lambda))-\epsilon M_i$, is shown against the distance from the wall for $n = 1$, $\lambda = 0$, $\lambda = 0.5$, $nt = \pi/2$, and $c = 0.5$. It is clear from the figure that at certain times the flow near to the surface is in negative x-direction, while the flow in the main stream is alsways in positive x-direction. Consequently, certain members of the class of transient velocity profiles are of a "separation type" and the point of separation for viscoelastic fluids is nearer to the surface than that of Newtonian fluids. This figure also shows that the velocity field in the boundary layer for viscoelastic fluids is more than that of Newtonian fluids.

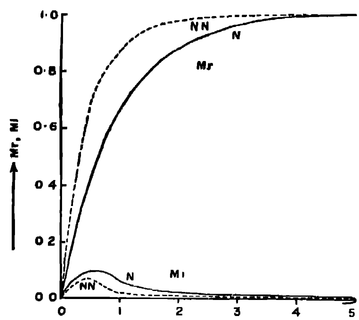

Figure 3
Fluctuating parts of velocity profiles (n = 1)

——→ N : Newtonian ($\lambda = 0$)

- - - → NN . Non-Newtonian ($\lambda = 1/2$)

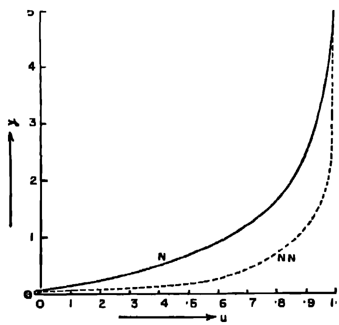

Figure 4
Transient velocity profiles $u = 1 - \exp\left\{-\frac{y}{1-\lambda}\right\} - \epsilon\, Mi$

——→ N . Newtonian ($\lambda = 0$)

- - - → NN : Non-Newtonian ($\lambda = 1/2$)

ACKNOWLEDGEMENT

The author expresses his sincere thanks to the referee for his valuable comments and to Dr. S. R. Mukherjee, Institute of Technology, Banaras Hindu University, for his kind suggestions and guidance in the preparation of this paper.

REFERENCES

Beard D. W. ' Walters K. 1964 *Proc. Camb. Phil. Soc.*, **60**, 667.
Lighthill M. J. 1954 *Proc. Roy. Soc.*, **224a**, 1.
Michael C. W. & Bird R. B. 1962 *A. I. Ch. E.*, **8**, 378.
Rath R. S. & Mishra B. 1969 *The Math. Educ.* **3**, No. 4, 124.
Stuart J. T. 1955 *Proc. Roy. Soc.* **231a**, 116.